Complexity of Reasoning with Cardinality Minimality Conditions

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Structure of the Talk

1. The CardMinSat problem,
2. Preliminaries (propositional logic, constraint languages),
3. Complexity Results,
4. Concluding Remarks.

The CardMinSat Problem

Problem: CardMinSat
Input: A propositional formula ϕ and a variable $x \in \text{var}(\phi)$.
Question: Is x true in a cardinality-minimal model of ϕ?

Cardinality-minimal model: a model with minimal number of variables assigned to 1.

The CardMinSat Problem

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CardMinSat is complete for the class $\mathbf{\Theta}_2^{\mathrm{P}}$ (Wagner 1988; C., Pichler, Woltran 2018).

$\mathbf{\Theta}_2^{\mathrm{P}} = \mathsf{P}^{\mathsf{NP}[O(log\ n)]}$ = polynomial with a logarithmic number of calls to an NP-oracle.

$\mathsf{NP} \subseteq \mathbf{\Theta}_2^{\mathrm{P}} \subseteq \mathsf{P}^{\mathsf{NP}} \subseteq \mathsf{NP}^{\mathsf{NP}}$

The CardMinSat Problem - relevance to AI

Many AI-related reasoning problems use some notion of minimality. For instance

- Belief revision.
  Principle: minimal change
- Abduction.
  Minimal explanations, relevance problems.

When cardinality-minimality is chosen as minimality criterion, all these problems are closely related to CardMinSat.

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When cardinality-minimality is chosen as minimality criterion, all these problems are closely related to CardMinSat.

Conducting a (more) detailed complexity analysis of CardMinSat can therefore advance our understanding of these problems' complexity.

The CardMinSat Problem - detailed complexity analysis

Consider CardMinSat in fragments of propositional logic in Schaefer's framework.

This will cover well-known fragments such as Horn, Krom, affine, and many more.

Relational Algebra

Definition

- A logical relation of arity k is a relation $R \subseteq \{0,1\}^k$.

Example (logical relation)

$N = \{0,1\}^3 \setminus \{\, 000, 111 \,\}$, a relation of arity 3.

Relational Algebra

Definition

- A logical relation of arity k is a relation $R \subseteq \{0,1\}^k$.
- A constraint C over R is a formula $C = R(x_1, \ldots, x_k)$, where $x_i' s$ are variables.

Example (constraint)

$C = N(x_1, x_2, x_3)$

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- An assignment σ to the x_i's satisfies C if $(\sigma(x_1), \ldots, \sigma(x_k)) \in R$.

Example (assignment satisfies constraint)

$\{011\} \models N(x_1, x_2, x_3), \quad \{111\} \not\models N(x_1, x_2, x_3)$.

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- A constraint language Γ is a finite set of logical relations.

Example (constraint language)

$\Gamma = \{N, E, D\} = \{\{0,1\}^3 \setminus \{\,000, 111\,\}, \{00, 11\}, \{10, 01\}\}$.

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- An assignment σ to the x_i's satisfies C if $(\sigma(x_1), \dots, \sigma(x_k)) \in R$.
- A constraint language Γ is a finite set of logical relations.
- A Γ-formula is a finite conjunction of constraints over relations in Γ.

Example (Γ-formula)

$N(x,y,z) \wedge E(x,y) \wedge D(x,z)$ where $N, E, D \in \Gamma$.

Relational Algebra

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- A constraint C over R is a formula $C = R(x_1, \dots, x_k)$, where $x_i's$ are variables.
- An assignment σ to the x_i's satisfies C if $(\sigma(x_1), \dots, \sigma(x_k)) \in R$.
- A constraint language Γ is a finite set of logical relations.
- A Γ-formula is a finite conjunction of constraints over relations in Γ.
- An assignment σ satisfies an Γ-formula ϕ if σ satisfies every constraint in ϕ.

Example (assignment satisfies Γ-formula)

$\{001\} \models N(x,y,z) \wedge E(x,y) \wedge D(x,z)$.

Problems in Schaefer's framework

Problem:	SAT(Γ)
Input:	A Γ-formula ϕ
Question:	Is ϕ satisfiable?

Problem:	CardMinSat(Γ)
Input:	A Γ-formula ϕ and a variable $x \in \mathrm{var}(\phi)$
Question:	Is x true in a cardinality-minimal model of ϕ?

Specific Constraint Languages

A k-ary relation R is represented by a formula ϕ in CNF if ϕ is a formula over k distinct variables $x_1, \ldots, x_k$ and $\phi \equiv R(x_1, \ldots, x_k)$.

Example

- $E(x, y) \equiv (x + y) = 0 \equiv (\bar{x} \vee y) \wedge (x \vee \bar{y})$
- $D(x, y) \equiv (x + y = 1) \equiv (x \vee y) \wedge (\bar{x} \vee \bar{y})$
- $N(x, y, z) \equiv (x \vee y \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$

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A relation R is

- Horn, dualHorn, Krom, 0/1-valid, if ϕ is so.
- Krom, if ϕ is a 2-CNF formula.
- Affine, if ϕ is a conjunction of linear equations (over $\{0, 1\}$).
- Width-2-affine, if ϕ is a conjunction of linear equations of size two, i.e., $(x \neq y), (x = y)$.

Specific Constraint Languages

Example

- $E(x,y) \equiv (x+y=0) \equiv (\bar{x} \vee y) \wedge (x \vee \bar{y})$
  E is width-2-affine, Krom, Horn, 0 and 1-valid.

Specific Constraint Languages

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- $D(x, y) \equiv (x + y = 1) \equiv (x \vee y) \wedge (\bar{x} \vee \bar{y})$
  D is width-2-affine, Krom, but not Horn $(01, 10$ are models but 00 is not$)$.

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- $D(x, y) \equiv (x + y = 1) \equiv (x \vee y) \wedge (\bar{x} \vee \bar{y})$
  D is width-2-affine, Krom, but not Horn ($01, 10$ are models but 00 is not).
- $N(x, y, z) \equiv (x \vee y \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$
  N is not affine, not Krom, not Horn

Schaefer's Dichotomy Theorem (STOC 1978)

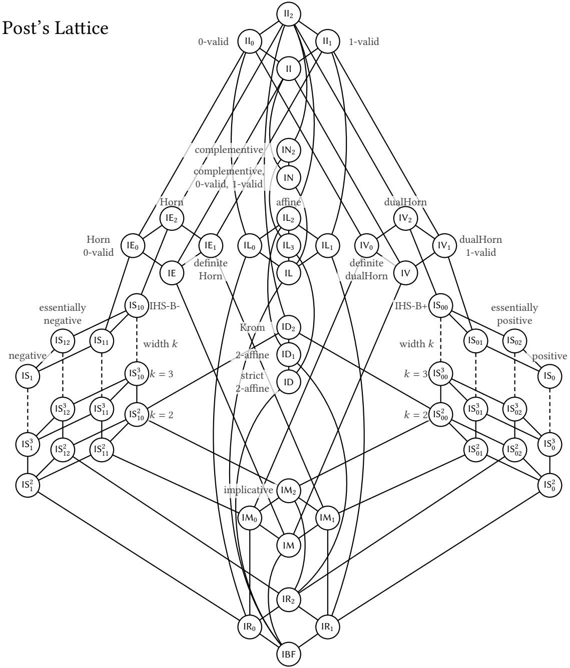
Theorem

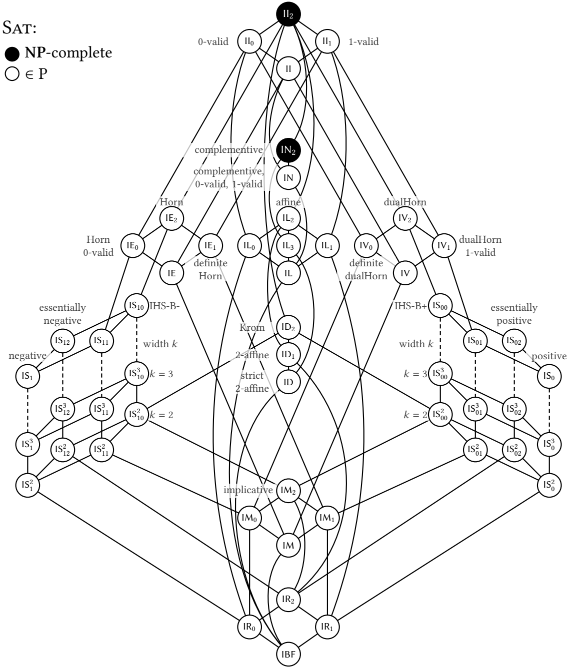
SAT(Γ) is in P if Γ is Horn, dual Horn, Krom, affine or 0- or 1-valid, and NP-complete otherwise.

A note on proof methods

- In 1978 Schaefer's Theorem was proven via many case distinctions.
- In the late 90's tools from universal algebra simplified the proof significantly (Jeavons 1998).
  - The expressivity of a relation is characterized by the closure properties of its set of models.
  - Closure functions sets are clones, described by Post's lattice.
  - Complexity results for $\mathrm{SAT}(\Gamma)$ can be obtained through a systematic examination of Post's lattice.

Post's Lattice




SAT:
NP-complete
∈ P
II2
II0
II1
0-valid
1-valid
II
complementive
IN2
complementive,
0-valid, 1-valid
IN
Horn
affine
dualHorn
IE2
IL2
IV2
Horn
0-valid
IE0
IE1
IL0
IL3
IL1
IV0
IV1
dualHorn
1-valid
IE
definite
Horn
IL
definite
dualHorn
IV
IS10
IHS-B-
IHS-B+
IS00
essentially
negative
essentially
positive
IS12
IS11
Krom
ID2
IS01
IS02
width k
width k
negative
2-affine
ID1
positive
IS1
IS0
IS10^3
k = 3
strict
2-affine
ID
k = 3
IS00^3
IS12^3
IS11^3
IS10^2
k = 2
k = 2
IS00^2
IS01^3
IS02^3
IS1^3
IS12^2
IS11^2
IS01^2
IS02^2
IS0^3
IS1^2
implicative
IM2
IS0^2
IM0
IM1
IM
IR2
IR0
IR1
IBF

New Results

Main Theorem

Problem:	CardMinSat(Γ)
Input:	A Γ-formula ϕ and a variable $x \in \text{var}(\phi)$
Question:	Is x true in a cardinality-minimal model of ϕ?

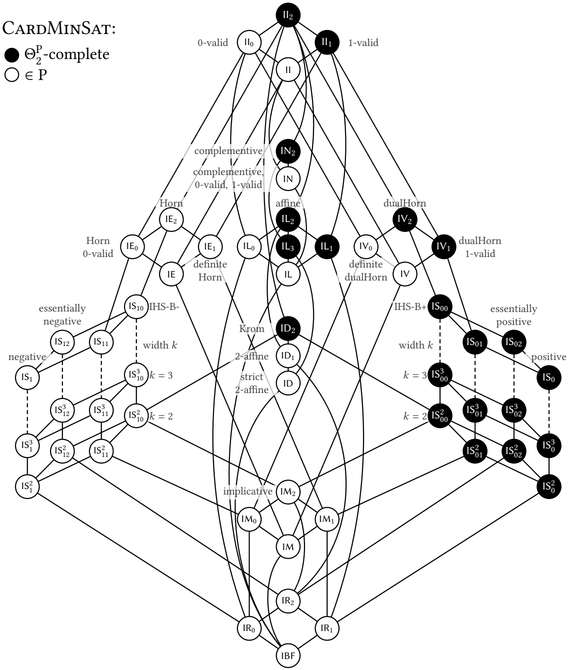
Theorem

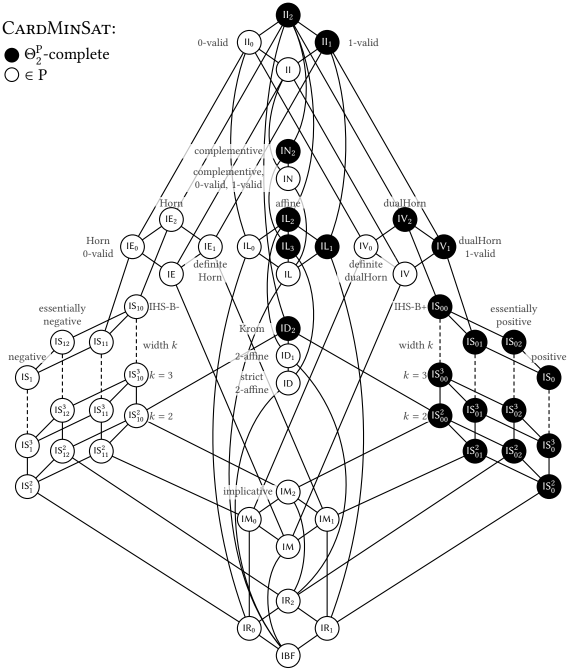
CardMinSat(Γ) in P if Γ is width-2-affine or Horn or 0-valid,
and $\mathbf{\Theta}_2^{\mathrm{P}}$-complete otherwise.

CardMinSat:

● Θ_2^P-complete

○ $\in$ P




CardMinSat:
Θ_2^{P}-complete
$\in$ P
II_2
0-valid
II_0
II_1
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II
complementive
IN_2
complementive, 0-valid, 1-valid
IN
Horn
affine
dualHorn
IE_2
IL_2
IV_2
Horn 0-valid
IE_0
IE_1
IL_0
IL_3
IL_1
IV_0
IV_1
dualHorn 1-valid
IE
definite Horn
IL
definite dualHorn
IV
essentially negative
IS_{10}
IHS-B-
Krom
ID_2
IHS-B+
IS_{00}
essentially positive
IS_{12}
IS_{11}
width k
ID_1
width k
IS_{01}
IS_{02}
negative
2-affine
positive
IS_1
IS_{10}^3
$k = 3$
strict 2-affine
ID
$k = 3$
IS_{00}^3
IS_0
IS_{12}^3
IS_{11}^3
IS_{10}^2
$k = 2$
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IS_{00}^2
IS_{01}^3
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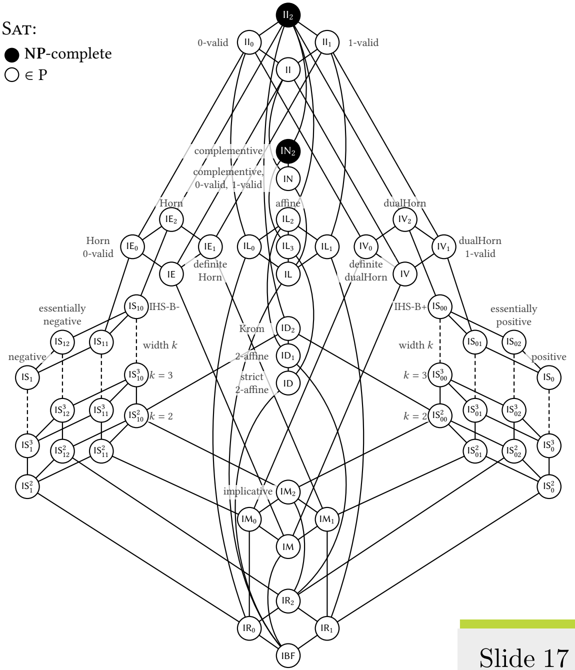
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IM
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IR_1
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A note on proof methods

For CardMinSat, the initial algebraic tools are not applicable, and we use advanced algebraic tools (Schnoor&Schnoor 2008, Lagerkvist 2014).

Application to Abduction

A propositional abduction problem $\mathcal{P} = (V, H, M, T)$, where:

- V is a finite set of variables,
- $H \subseteq V$ is the set of hypotheses,
- $M \subseteq V$ is the set of manifestations and
- T is a consistent theory in the form of a propositional formula.

A set $\mathcal{S} \subseteq H$ is a solution (also called explanation) to $\mathcal{P}$ if $T \cup \mathcal{S}$ is consistent and $T \cup \mathcal{S} \models M$ holds.

Problem:	Card-min-Relevance
Input:	$\mathcal{P} = (V, H, M, T)$ and hypothesis $h \in H$.
Question:	Is h relevant, i.e., does $\mathcal{P}$ admit a cardinality-minimal solution $\mathcal{S}$ such that $h \in \mathcal{S}$?

Complexity of Card-min-Relevance

Problem:	Card-min-Relevance
Input:	PAP $\mathcal{P} = (V, H, M, T)$ and hypothesis $h \in H$.
Question:	Is h relevant, i.e., does $\mathcal{P}$ admit a cardinality-minimal solution $\mathcal{S}$ such that $h \in \mathcal{S}$?

The Card-min-Relevance problem is:

- Θ_3^{P}-complete in its full generality (Eiter and Gottlob, 1995)
- Θ_2^{P}-complete in the Horn case (Eiter and Gottlob, 1995)
- Θ_2^{P}-complete in the Krom case (C., Pichler, Woltran, 2018)

Here we prove that the Card-min-Relevance problem is Θ_2^{P}-complete in the affine case, by a reduction from CardMinSat($\{x \oplus y \oplus z\}$)

Conclusion

- The result is still a dichotomy, only membership in P and $\mathbf{\Theta}_2^{\mathrm{P}}$-completeness arise, nothing in between.

Future Work

- Address systematic complexity classifications for the related problems from belief revision and abduction.
- Consider parametrized complexity. For Abduction a lot is already done here, but the abduction relevance problem based on cardinality-minimality is a blind spot, even in classic complexity. CardMinSat will help advance here.

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☺ Thanks! ☺

And thanks to our sponsor: CIRM, Research in pairs 1886.